Question 1: (32 points) Circle the most correct answer in the following

1) The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n+1}{n} = 2 \neq 0$

div. by nth term test.

a) Converges absolutely
b) Converges conditionally
c) Diverges by n-th term test

2) One of the following is true
a) If $\sum_{n=0}^{\infty}|a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges
b) If $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty}|a_n|$ converges
c) If $\sum_{n=0}^{\infty}|a_n|$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges

3) The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

P-series, $P = \frac{1}{2} \leq 1$.
$\Rightarrow$ Conv. Cond.

a) Converges absolutely
b) Converges conditionally
c) Diverges by n-th term test

4) The area under parametric curve: : $x = t\,, y = 2t,\ 1 \leq t \leq 3$ is
a) 6
b) 9
c) 8

$A = \int_1^3 y\,dx = \int_1^3 2t\,dt = t^2\Big|_1^3 = 9 - 1 = 8.$

$(x_1, y_1)\quad (x_2, y_2)$

5) The parameterization of the line segment from $(0,2)$ to $(1,3)$ is
a) $x = t\,, y = 2 + t\,, 0 \leq t \leq 1$
b) $x = t\,, y = 2 + t\,, 2 \leq t \leq 3$
c) $x = t - 2\,, y = t\,, 0 \leq t \leq 1$

slope $= \frac{3-2}{1-0} = \boxed{1}$.
$y = 2 + (1)(x - 0)$.
$y = 2 + x$
suppose $t = x \Rightarrow y = 2 + t$, $0 \leq t \leq 1$ (since $0 \leq x \leq 1$).

6) The interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{3^n x^n}{n!}$
a) $(-1,1)$
b) $(-2,2)$
c) $(-\infty, \infty)$

by Ratio test $\Rightarrow$
$\lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = 0 < 1$
for all x, so $IC = (-\infty, \infty)$.

7) If : $x = 2t\,, y = t^2, then\ \frac{d^2y}{dx^2}\ at\ t = \frac{1}{4}$ is
a) 1
b) $\frac{1}{4}$
c) $\frac{1}{2}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = \boxed{t}.$

$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \boxed{\frac{1}{2}}$

8) The Binomial series of $f(x) = \frac{1}{\sqrt{1+x}}$ *is* $= (1+x)^{-\frac{1}{2}}$

a) $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots$

b) $1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots$

c) $1 - \frac{x}{2} - \frac{3x^2}{8} - \frac{5x^3}{16} - \cdots$

$= 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} x^k$

$= 1 + -\frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2} x^2 + \cdots$

$= 1 - \frac{x}{2} + \frac{3}{8}x^2 - \cdots$

9) The power series $\sum_{n=1}^{\infty} n^n (x-2)^n$ converges absolutely at

a) The center $x = 2$

b) $-2 < x < 2$

c) $1 < x < 3$

Root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|n^n (x-2)^n|}$

$= \lim_{n\to\infty} n|x-2| = \infty > 1$

div. $\forall x$ except at the center. $x = 2$.

10) The graph of the parametric curve of $x = 2cost$ and $y = 2sint$ for $0 \le t \le \pi$ is:

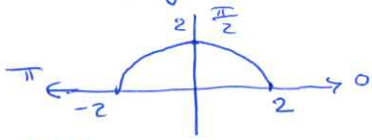
a) Circle with center (0,0) and radius =2

b) Half circle with center (0,0) and radius =2

c) Lower down of hyperbola

$x^2 + y^2 = 4(\cos^2 t + \sin^2 t)$.

$x^2 + y^2 = 4$.



11) The Maclaurin series of $f(x) = x^2 sinx$ is

a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!}$

b) $\sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$x^2 \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}$

12) The tangent line for the parametric curve whose parametric equations are: $x = t, y = \sqrt{t}$ *at* $t = \frac{1}{4}$ *is*

a) $y = x + 1$

b) $y = x + \frac{1}{4}$

c) $y = x + \frac{1}{2}$

at $t = \frac{1}{4} \Rightarrow x = \frac{1}{4}, y = \frac{1}{2} \Rightarrow (\frac{1}{4}, \frac{1}{2})$.

$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{t}}}{1} \Big|_{t=\frac{1}{4}} = \frac{1}{2 \cdot \frac{1}{2}} = \boxed{1}$.

$y = \frac{1}{2} + (1)(x - \frac{1}{4}) = \frac{1}{2} + x - \frac{1}{4} = \boxed{x + \frac{1}{4}}$

13) The length of the parametric curve: $x = 2t,\ y = \sqrt{5}\,t, 0 \le t \le 4$ is

a) 4

b) 12

c) 36

$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^4 \sqrt{4+5}\, dt = 3t \Big|_0^4 = 3(4-0)$

$= 12$.

14) If the first four terms are used to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ then the error satisfies

$= \boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}} + \frac{1}{5} - \frac{1}{6} + \cdots$

the first neglected term is Positive & $\frac{1}{5} = \frac{2}{10} = 0.2$

a) The error is positive and $|error| < 0.1$
b) The error is negative and $|error| < 0.2$
c) The error is positive and $|error| < 0.2$

15) For what values of x we can replace $\sin x$ *by* $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ So that $|Error| < 5 \times 10^{-4}$

$|Error| < |\text{the first neglected term}| < 5 \times 10^{-4}$.

$\left|\frac{x^9}{9!}\right| < 5 \times 10^{-4}$.

$|x| < \sqrt[9]{5 \times 9! \times 10^{-4}}$.

a) $|x| < \sqrt[7]{5 \times 9! \times 10^{-4}}$
b) $|x| < \sqrt[9]{5 \times 9! \times 10^{-4}}$
c) $|x| < 5 \times 9! \times 10^{-4}$

16) The Taylor's polynomial of order 2 for $f(x) = \ln x$ at $a = 1$ is

a) $(x-1) - \frac{1}{2}(x-1)^2$
b) $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$
c) $1 + (x-1) - \frac{1}{2}(x-1)^2$

$P_2 = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!}$

$= 0 + (x-1) - \frac{1}{2}(x-1)^2$.

$= (x-1) - \frac{1}{2}(x-1)^2$.

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Question 2: Use series to evaluate the limit

$$\lim_{x \to 0} \frac{x - tan^{-1}x}{x^3}$$

1 2

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

3

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{\cancel{x} - \cancel{x} + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots}{x^3}$$

1

$$= \lim_{x \to 0} \left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \cdots \right)$$

1

$$= \boxed{\frac{1}{3}}$$

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Question 3: Let

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

a) Find the interval of convergence
b) Find the series radius
c) For what values of x does the series converges absolutely
d) For what values of x does the series converges conditionally

The series converges abs. by root test if:

1 $$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

1 $$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(3x-2)^n}{n}\right|} < 1.$$

$$\lim_{n\to\infty} \left|\frac{3x-2}{\sqrt[n]{n}}\right| < 1$$

1 $$|3x-2| < 1$$

1 $$-1 < 3x-2 < 1$$

1 $$\frac{1}{3} < x < 1$$

at $x = \frac{1}{3} \Rightarrow$ 1 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ 1 conv. cond. 1 (P-series, $P \leq 1$).

at $x = 1 \Rightarrow$ 1 $\sum_{n=1}^{\infty} \frac{1}{n}$ 1 diverges 1 (harmonic series). P-series $P \leq 1$.

2 a) $\frac{1}{3} \leq x < 1$

1 b) $R = \frac{1}{3}$

1 c) $\frac{1}{3} < x < 1$

1 d) $x = \frac{1}{3}$

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Question 4: Find the Taylor series generated by $f(x) = 3^{-x}$ *at* $x = 1$. (Write the final answer using sigma notation).

$n=0$ ① $f(1) = 3^{-1} = \frac{1}{3}$.

① $f'(x) = -\ln 3 \; 3^{-x}$ $\Rightarrow$ ① $f'(1) = \frac{-\ln 3}{3}$

① $f''(x) = (\ln 3)^2 \, 3^{-x}$ $\Rightarrow$ ① $f''(1) = \frac{(\ln 3)^2}{3}$

① $f'''(x) = -(\ln 3)^3 \, 3^{-x}$ $\Rightarrow$ ① $f'''(1) = -\frac{(\ln 3)^3}{3}$.

$\vdots$

② $f^n(x) = (-1)^n \frac{(\ln 3)^n}{3}$.

The Taylor Series is ① $\sum_{n=0}^{\infty} \frac{f^n(1)(x-1)^n}{n!}$

6 $= \sum_{n=0}^{\infty} (-1)^n \frac{(\ln 3)^n (x-1)^n}{3\, n!}$ ① ① ① ① ① ①

6

Question 5: Let the parametric equations:

$$x = 3t\,, \qquad y = 9t^2 \quad , \quad t \geq 0.$$

2 a) Find a Cartesian equation.

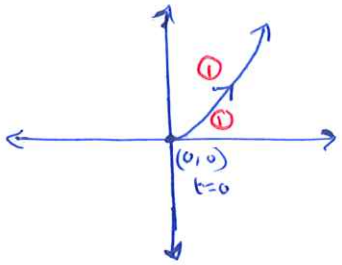
2 b) Graph the Cartesian equation and show the direction of motion

2 c) What are the initial and terminal points

a $x = 3t \Rightarrow x^2 = 9t^2 \quad t \geq 0.$

1 1 $\boxed{y = x^2}$

b



c) $\underline{\text{I.P}}$ at $t=0 \Rightarrow x=0$ & $y=0$ 1 $(0,0)$.

No terminal point 1

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Question 6: Estimate the integral $\int_0^1 e^{-x}dx$ error of magnitude less than 0.1

1 2

$$e^{-x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

1 $$\int_0^1 e^{-x}\,dx = \int_0^1 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right) dx.$$

2 $$= x - \frac{x^2}{2} + \frac{x^3}{3(2)} - \frac{x^4}{4(6)} + \frac{x^5}{5(4!)} - \dots \Big|_0^1$$

1 $$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \dots$$

1 $|\text{Error}| < \underbrace{|\text{the first neglected term}|}_{\left|-\frac{1}{24}\right| < \frac{1}{10}} < \frac{1}{10}$

So $\int_0^1 e^{-x}\,dx \approx 1 - \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$ 1 1